

Termination for the Direct Sum of Left-Linear Term Rewriting Systems - Preliminary Draft* -

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Abstract

The direct sum of two term rewriting systems is the union of systems having disjoint sets of function symbols. It is shown that two term rewriting systems both are left-linear and complete if and only if the direct sum of these systems is so.

1. Introduction

An important concern in building algebraic specifications is their hierarchical or modular structure. The same holds for term rewriting systems [1] which can be viewed as implementations of equational algebraic specifications. Specifically, it is of obvious interest to determine which

*This paper is an abbreviated version of the *IEICE technical report COMP88-30*, July 1988. Now we are preparing a final version for submission based on this draft.

properties of term rewriting systems have a *modular* character, where we call a property *modular* if its validity for a term rewriting system, hierarchically composed of some smaller term rewriting systems, can be inferred from the validity of that property for the constituent term rewriting systems. Naturally, the first step in such an investigation considers the most basic properties of term rewriting systems: confluence, termination, unique normal form property, and similar fundamental properties as well as combinations thereof.

As to the modular structure of term rewriting systems, it is again natural to consider as a start the most simple way that term rewriting systems can be combined to form a larger term rewriting system: namely, as a disjoint sum. This means that the alphabets of the term rewriting systems to be combined are disjoint, and that the rewriting rules of the sum term rewriting system are the rules of the summand term rewriting systems together. (Without the disjointness requirement the situation is even more complicated - see for some results in this direction: Dershowitz [2], Toyama [10].) A disjoint union of two term rewriting systems R_0 and R_1 is called in our paper a direct sum, notation $R_0 \oplus R_1$.

Another simplifying assumption that we will make, is that R_0, R_1 are homogeneous term rewriting systems, i.e. their signature is one-sorted (as opposed to the many sorted or heterogeneous case; for results about direct sums of heterogeneous term rewriting systems, see Ganzinger and Giegerich [3].)

The first result in this setting is due to Toyama [8], where it is proven that confluence is a modular property. (I.e. R_0 and R_1 are confluent iff $R_0 \oplus R_1$ is so. Here \Leftarrow is trivial; \Rightarrow is what we are interested in.) To appreciate the non-triviality of this fact, it may be contrasted with the fact that another fundamental property, termination, is *not* modular, as the following simple counterexample in Toyama [9] shows:

$$R_0 \quad \left\{ \begin{array}{l} F(0, 1, x) \triangleright F(x, x, x) \end{array} \right.$$

$$R_1 \quad \left\{ \begin{array}{l} g(x, y) \triangleright x \\ g(x, y) \triangleright y \end{array} \right.$$

It is trivial that R_0 and R_1 are terminating. However, $R_0 \oplus R_1$ is not terminating, because $R_0 \oplus R_1$ has the infinite reduction sequence:

$$\begin{aligned} F(g(0, 1), g(0, 1), g(0, 1)) &\rightarrow F(0, g(0, 1), g(0, 1)) \rightarrow F(0, 1, g(0, 1)) \\ &\rightarrow F(g(0, 1), g(0, 1), g(0, 1)) \rightarrow \dots \end{aligned}$$

The above counterexample uses a non-confluent term rewriting system R_1 . A more complicated counterexample to the modularity of *termination*, involving only confluent term rewriting

systems, was given by Klop and Barendregt [4] (for ground terms only; for some improved versions, holding for open terms as well, and even using term rewriting systems which are *irreducible*, see Toyama [9]). This means that the important property of *completeness* of term rewriting systems (a term rewriting system is complete iff it is both confluent and terminating) is not modular, i.e. there are complete term rewriting systems R_0, R_1 such that $R_0 \oplus R_1$ is not complete (in fact, not terminating; confluence of $R_0 \oplus R_1$ is ensured by the theorem in Toyama [8]). This counterexample, however, uses non-left-linear term rewriting systems.

The point of the present paper is that left-linearity is essential; if we restrict ourselves to left-linear term rewriting systems, then completeness is modular. Thus we prove: If R_0 and R_1 are left-linear (meaning that the rewriting rules have no repeated variables in their left-hand-sides), then R_0 and R_1 are complete iff $R_0 \oplus R_1$ is so. As left-linearity is a property which is so easily checked, and many equational algebraic specifications can be given by term rewriting systems which are left-linear, we feel that this result is worth while.

The proof, however, is rather intricate and not easily digested. A crucial element in the proof, and in general in the way that the summand term rewriting systems interact, is how terms may *collapse* to a subterm. The problem is that this collapsing behavior may exhibit a *nondeterministic* feature, which is caused by ambiguities among the rewriting rules. We hope that the present paper is of value not only because it establishes a result that in itself is simple enough, but also because of the analysis necessary for the proof which gives a kind of structure theory for disjoint combinations of term rewriting systems and which may be of relevance in other, similar, studies.

Regarding the question of modular properties in the present simple set-up, we mention the recent results by Rusinowitch [7] and Middeldorp [5]; these papers, together, contain a complete analysis of the cases in which termination for $R_0 \oplus R_1$ may be concluded from termination of R_0, R_1 , depending on the distribution among R_0, R_1 of so-called collapsing and duplicating rules.

Another useful fact is established in Middeldorp [6], where it is proven that the *unique normal form property* is a modular property.

2. Notations and Definitions

Assuming that the reader is familiar with the basic concepts and notations concerning term rewriting systems in [1,8], we briefly explain notations and definitions for the following discussions.

Let F be a set of function symbols, and let V be a set of variable symbols. By $T(F, V)$, we denote the set of terms constructed from F and V .

A term rewriting system R is a set of rewriting rules $M \triangleright N$, where M and N are terms

disjoint function symbols [8].

In this paper, we assume that two disjoint systems R_0 on $T(F_0, V)$ and R_1 on $T(F_1, V)$ both are left-linear and complete. Then we shall prove that the direct sum system $R_0 \oplus R_1$ on $T(F_0 \cup F_1, V)$ is terminating. From here on the notation \rightarrow represents the reduction relation on $R_0 \oplus R_1$.

Lemma 2.1. $R_0 \oplus R_1$ is weakly normalizing, i.e., every term M has a normal form (denoted by $M \downarrow$).

Proof. Since R_0 and R_1 are terminating, M can be reduced into $M \downarrow$ through innermost reduction. \square

The identity of terms of $T(F_0 \cup F_1, V)$ (or syntactical equality) is denoted by \equiv . $\xrightarrow{*}$ is the transitive reflexive closure of \rightarrow , $\xrightarrow{+}$ is the transitive closure of \rightarrow , $\xrightarrow{\equiv}$ is the reflexive closure of \rightarrow , and $=$ is the equivalence relation generated by \rightarrow (i.e., the transitive reflexive symmetric closure of \rightarrow). \xrightarrow{m} denotes a reduction of m ($m \geq 0$) steps.

Definition. A *root* is a mapping from $T(F_0 \cup F_1, V)$ to $F_0 \cup F_1 \cup V$ as follows: For $M \in T(F_0 \cup F_1, V)$,

$$root(M) = \begin{cases} f & \text{if } M \equiv f(M_1, \dots, M_n), \\ M & \text{if } M \text{ is a constant or a variable.} \end{cases}$$

Definition. Let $M \equiv C[B_1, \dots, B_n] \in T(F_0 \cup F_1, V)$ and $C \neq \square$. Then write $M \equiv C[B_1, \dots, B_n]$ if $C[_, \dots, _]$ is a context on F_d and $\forall i, root(B_i) \in F_{\bar{d}}$ ($d \in \{0, 1\}$ and $\bar{d} = 1 - d$). Then the set $S(M)$ of the special subterms of M is inductively defined as follows:

$$S(M) = \begin{cases} \{M\} & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \cup_i S(B_i) \cup \{M\} & \text{if } M \equiv C[B_1, \dots, B_n] \ (n > 0). \end{cases}$$

The set of the special subterms having the root symbol in F_d is denoted by $S_d(M) = \{N \mid N \in S(M) \text{ and } root(N) \in F_d\}$.

Let $M \equiv C[B_1, \dots, B_n]$ and $M \xrightarrow{A} N$ (i.e., N results from M by contracting the redex occurrence A). If the redex occurrence A occurs in some B_j , then we write $M \xrightarrow{i} N$; otherwise $M \xrightarrow{o} N$. Here, \xrightarrow{i} and \xrightarrow{o} are called an inner and an outer reduction, respectively.

Definition. For a term $M \in T(F_0 \cup F_1, V)$, the rank of layers of contexts on F_0 and F_1 in M is inductively defined as follows:

$$rank(M) = \begin{cases} 1 & \text{if } M \in T(F_d, V) \ (d = 0 \text{ or } 1), \\ \max_i \{rank(B_i)\} + 1 & \text{if } M \equiv C[B_1, \dots, B_n] \ (n > 0). \end{cases}$$

Lemma 2.2. If $M \rightarrow N$ then $\text{rank}(M) \geq \text{rank}(N)$.

Proof. It is easily obtained from the definitions of the direct sum. \square

Lemma 2.3. Let $M \xrightarrow{*} N$ and $\text{root}(M), \text{root}(N) \in F_d$. Then there exists a reduction $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \cdots \rightarrow M_n \equiv N$ ($n \geq 0$) such that $\text{root}(M_i) \in F_d$ for any i .

Proof. Let $M \xrightarrow{k} N$ ($k \geq 0$). We will prove the lemma by induction on k . The case $k = 0$ is trivial. Let $M \rightarrow M' \xrightarrow{k-1} N$ ($k > 0$). If $\text{root}(M') \in F_d$ then the lemma holds by the induction hypothesis. If $\text{root}(M') \in F_{\bar{d}}$ then there exists a context $C[\]$ with $\text{root} \in F_d$ such that $M \equiv C[M']$ and $C[\] \rightarrow \square$. Thus, we can obtain a reduction $M \equiv C[M'] \xrightarrow{*} C[N] \rightarrow N$ in which all terms have root symbols in F_d . \square

The set of terms in the reduction graph of M is denoted by $G(M) = \{N \mid M \xrightarrow{*} N\}$. The set of terms having the root symbol in F_d is denoted by $G_d(M) = \{N \mid N \in G(M) \text{ and } \text{root}(N) \in F_d\}$.

Definition. A term M is erasable iff $M \xrightarrow{*} x$ for some $x \in V$.

From now on we assume that every term $M \in T(F_0 \cup F_1, V)$ has only x as variable occurrences, unless it is stated otherwise. Since $R_0 \oplus R_1$ is left-linear, this variable convention may be assumed in the following discussions without loss of generality. If we need fresh variable symbols not in terms, we use z, z_1, z_2, \dots .

3. Essential Subterms

In this section we introduce the concept of the essential subterms. We first prove the following property:

$$\forall N \in G_d(M) \exists P \in S_d(M), M \xrightarrow{*} P \xrightarrow{*} N.$$

Lemma 3.1. Let $M \rightarrow N$ and $Q \in S_d(N)$. Then, there exists some $P \in S_d(M)$ such that $P \equiv Q$.

Proof. We will prove the lemma by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial. Assume the lemma for $\text{rank}(M) < k$ ($k > 1$), then we will show the case $\text{rank}(M) = k$. Let $M \equiv C[M_1, \dots, M_n]$ ($n > 0$) and $M \xrightarrow{A} N$.

$$\text{Case 1. } M \equiv C[M_1, \dots, M_r, \dots, M_n] \xrightarrow{A} N \equiv M_r.$$

Then $S_d(N) \subseteq S_d(M)$.

Case 2. $M \equiv C[M_1, \dots, M_n] \xrightarrow{A}_o N \equiv C'[M_{i_1}, \dots, M_{i_p}]$ ($1 \leq i_j \leq n$).

If $\text{root}(M) \in F_d$ then

$$S_d(M) = \{M\} \cup \bigcup_i S_d(M_i),$$

$$S_d(N) = \{N\} \cup \bigcup_j S_d(M_{i_j}).$$

Thus the lemma holds since $\bigcup_j S_d(M_{i_j}) \subseteq \bigcup_i S_d(M_i)$, and $M \rightarrow N$.

If $\text{root}(M) \in F_d$ then $S_d(N) = \bigcup_j S_d(M_{i_j}) \subseteq \bigcup_i S_d(M_i) = S_d(M)$.

Case 3. $M \equiv C[M_1, \dots, M_r, \dots, M_n] \xrightarrow{A}_i N \equiv C[M_1, \dots, M'_r, \dots, M_n]$ where $M_r \xrightarrow{A} M'_r$.

If $\text{root}(M) \in F_d$ then

$$S_d(M) = \{M\} \cup S_d(M_r) \cup \bigcup_{i \neq r} S_d(M_i),$$

$$S_d(N) \subseteq \{N\} \cup S_d(M'_r) \cup \bigcup_{i \neq r} S_d(M_i).$$

If $\text{root}(M) \in F_d$ then

$$S_d(M) = S_d(M_r) \cup \bigcup_{i \neq r} S_d(M_i),$$

$$S_d(N) = S_d(M'_r) \cup \bigcup_{i \neq r} S_d(M_i).$$

By the induction hypothesis, $\forall Q \in S_d(M'_r) \exists P \in S_d(M_r), P \equiv Q$ for the both $\text{root}(M) \in F_d$ and $\text{root}(M) \in F_d$. Thus the lemma holds. \square

R_e consists of the single rule $e(x) \triangleright x$. \rightarrow_e denotes the reduction relation of R_e , and $\rightarrow_{e'}$ denotes the reduction relation of $R_e \oplus (R_0 \oplus R_1)$ such that if $C[e(P)] \xrightarrow{A}_{e'} N$ then the redex occurrence Δ does not occur in P . It is easy to show the confluence property of $\rightarrow_{e'}$.

From here on, $C[e(P_1), \dots, e(P_p)]$ denotes a term such that $C[P_1, \dots, P_p] \in T(F_0 \cup F_1, V)$, i.e., C and P_i contain no e .

Lemma 3.2. Let $C[e(P_1), \dots, e(P_{i-1}), e(P_i), e(P_{i+1}), \dots, e(P_p)] \xrightarrow{k}_{e'} e(P_i)$. Then $C[P_1, \dots, P_{i-1}, e(P_i), P_{i+1}, \dots, P_p] \xrightarrow{k'}_{e'} e(P_i)$ ($k' \leq k$).

Proof. It is easily obtained from the definition and the left-linearity of the reduction $\rightarrow_{e'}$. \square

Let $M \equiv C[P] \in T(F_0 \cup F_1, V)$ be a term containing no function symbol e . Now, consider $C[e(P)]$ by replacing the occurrence P in M with $e(P)$. Assume $C[e(P)] \xrightarrow{*}_{e'} e(P)$. Then, by tracing the reduction path, we can also obtain the reduction $M \equiv C[P] \xrightarrow{*} P$ (denoted by $M \xrightarrow{*}_{pull} P$) under $R_0 \oplus R_1$. We say that the reduction $M \xrightarrow{*}_{pull} P$ pulls up the occurrence P from M .

Example 3.1. Consider the two systems R_0 and R_1 :

$$R_0 \quad \left\{ \begin{array}{l} F(x) \rightarrow G(x, x) \\ G(C, x) \rightarrow x \end{array} \right.$$

$$R_1 \quad \left\{ \begin{array}{l} h(x) \rightarrow x \end{array} \right.$$

Then we have the reduction:

$$F(e(h(C))) \xrightarrow{e'} G(e(h(C)), e(h(C))) \xrightarrow{e'} G(h(C), e(h(C))) \xrightarrow{e'} G(C, e(h(C))) \xrightarrow{e'} e(h(C)).$$

Hence $F(h(C)) \xrightarrow[pull]{*} h(C)$. However, we cannot obtain $F(z) \xrightarrow[pull]{*} z$. Thus, in general, we cannot obtain $C[z] \xrightarrow[pull]{*} z$ from $C[P] \xrightarrow[pull]{*} P$. \square

Lemma 3.3. Let $P \xrightarrow{*} Q$ and let $C[Q] \xrightarrow[pull]{*} Q$. Then $C[P] \xrightarrow[pull]{*} P$.

Proof. Let $M \equiv C[e(Q)] \xrightarrow{e'}^k e(Q)$. We will prove the lemma by induction on k . The case $k = 0$ is trivial. Let $M \equiv C[e(Q)] \xrightarrow{e'} C'[e(Q), \dots, e(Q), \dots, e(Q)] \xrightarrow{e'}^{k-1} e(Q)$. Then, from Lemma 3.2 we can obtain the following reduction:

$$C'[Q, \dots, e(Q), \dots, Q] \xrightarrow{e'}^{k'} e(Q) \quad (k' \leq k-1).$$

By using the induction hypothesis, $C'[Q, \dots, e(P), \dots, Q] \xrightarrow{e'}^* e(P)$. Therefore, we can obtain $C[e(P)] \xrightarrow{e'} C'[e(P), \dots, e(P), \dots, e(P)] \xrightarrow{e'}^* C'[Q, \dots, e(P), \dots, Q] \xrightarrow{e'}^* e(P)$ from $P \xrightarrow{*} Q$. \square

Lemma 3.4. $\forall N \in G_d(M) \exists P \in S_d(M), M \xrightarrow[pull]{*} P \xrightarrow{*} N$.

Proof. If $\text{root}(M) \in F_d$ then the above property is trivial by taking M as P . Thus we consider only the non trivial case of $\text{root}(M) \in F_{\bar{d}}$. Let $M \xrightarrow{k} N$. We will prove the lemma by induction on k . The case $k = 1$ is trivial since $M \equiv C[M_1, \dots, M_r, \dots, M_n] \rightarrow N \equiv M_r$ for some r (i.e., take $P \equiv M_r$). Assume the lemma for $k-1$. We will prove the case k . Let $M \rightarrow M' \xrightarrow{k-1} N$.

Case 1. $\text{root}(M') \in F_d$.

Then $M \equiv C[M_1, \dots, M_r, \dots, M_n] \rightarrow M' \equiv M_r$ for some r . Take $P \equiv M_r$.

Case 2. $\text{root}(M') \in F_{\bar{d}}$.

By using the induction hypothesis, $\exists P' \in S_d(M')$, $M' \xrightarrow[pull]{*} P' \xrightarrow{*} N$. Here, from Lemma 3.1, there exists some $P \in S_d(M)$ such that $P \xrightarrow{\equiv} P'$. We will consider the following two subcases:

Case 2.1. $P \rightarrow P'$. Then $M \equiv C[P] \rightarrow M' \equiv C[P']$. Thus, by using Lemma 3.3, $M \equiv C[P] \xrightarrow[pull]{*} P \rightarrow P' \xrightarrow{*} N$.

Case 2.2. $P \equiv P'$. Then, for some context $C'[\cdot, \dots, \cdot]$, $M \equiv C[P] \rightarrow M' \equiv C'[P, \dots, P, \dots, P]$ and $C'[P, \dots, e(P), \dots, P] \xrightarrow[e']{*} e(P)$. Therefore

$C[e(P)] \rightarrow C'[e(P), \dots, e(P), \dots, e(P)] \xrightarrow[e]{*} C'[P, \dots, e(P), \dots, P] \xrightarrow[e']{*} e(P)$. Thus $M \equiv C[P] \xrightarrow[pull]{*} P \xrightarrow{*} N$. \square

Now, we introduce the concept of the essential subterms. The set $E_d(M)$ of the essential subterms of the term $M \in T(F_0 \cup F_1, V)$ is defined as follows:

$$E_d(M) = \{P \mid M \xrightarrow[pull]{*} P \in S_d(M) \text{ and } \neg \exists Q \in S_d(M) [M \xrightarrow[pull]{*} Q \not\xrightarrow{*} P]\}.$$

The following lemmas are easily obtained from the definition of the essential subterms and Lemma 3.4.

Lemma 3.5. $\forall N \in G_d(M) \exists P \in E_d(M), P \xrightarrow{*} N$.

Lemma 3.6. $E_d(M) = \phi$ iff $G_d(M) = \phi$.

We say M is deterministic for d if $|E_d(M)| = 1$; M is nondeterministic for d if $|E_d(M)| \geq 2$. The following lemma plays an important role in the next section.

Lemma 3.7 If $root(M \downarrow) \in F_d$ then $|E_d(M)| = 1$, i.e., M is deterministic for d .

Proof. See Appendix in [11]. \square

4. Termination for the Direct Sum

In this section we will show that $R_0 \oplus R_1$ is terminating. Roughly speaking, termination is proven by showing that any infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ of $R_0 \oplus R_1$ can be translated into an infinite reduction $M'_0 \rightarrow M'_1 \rightarrow M'_2 \rightarrow \dots$ of R_d .

We first define the term $M^d \in T(F_d, V)$ for any term M and any d .

Definition. For any M and any d , $M^d \in T(F_d, V)$ is defined by induction on $\text{rank}(M)$:

- (1) $M^d \equiv M$ if $M \in T(F_d, V)$.
- (2) $M^d \equiv x$ if $E_d(M) = \phi$.
- (3) $M^d \equiv C[M_1^d, \dots, M_m^d]$ if $\text{root}(M) \in F_d$ and $M \equiv C[M_1, \dots, M_m]$ ($m > 0$).
- (4) $M^d \equiv P^d$ if $\text{root}(M) \in F_d$ and $E_d(M) = \{P\}$. Note that $\text{rank}(P) < \text{rank}(M)$.
- (5) $M^d \equiv C_1[C_2[\dots C_{p-1}[C_p[x]] \dots]]$ if $\text{root}(M) \in F_d$, $E_d(M) = \{P_1, \dots, P_p\}$ ($p > 1$), and every P_i^d is erasable. Here $P_i^d \equiv C_i[x] \xrightarrow[\text{pull}]{*} x$ ($i = 1, \dots, p$). Note that, for any i , $\text{rank}(P_i) < \text{rank}(M)$ and $M^d \xrightarrow{*} P_i^d$.
- (6) $M^d \equiv x$ if $\text{root}(M) \in F_d$, $|E_d(M)| \geq 2$, and not (5).

Note that M^d is not unique if a subterm of M^d is constructed with (5) in the above definition.

Lemma 4.1. $\text{root}(M \downarrow) \notin F_d$ iff $M^d \downarrow \equiv x$.

Proof. Instead of the lemma, we will prove the following claim:

Claim. If $\text{root}(M \downarrow) \notin F_d$ then $M^d \downarrow \equiv x$. If $\text{root}(M \downarrow) \in F_d$ and $M \downarrow \equiv \hat{C}[M_1, \dots, M_m]$ then $M^d \downarrow \equiv \hat{C}[x, \dots, x]$.

Proof of the Claim. We will prove the lemma by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial by the definition of M^d . Assume the lemma for $\text{rank}(M) < k$ ($k \geq 2$). Then we will prove the case $\text{rank}(M) = k$.

Case 1. $\text{root}(M) \in F_d$.

Let $M \equiv C[M_1, \dots, M_m]$. Then $M^d \equiv C[M_1^d, \dots, M_m^d]$. We may assume that $\text{root}(M_i \downarrow) \notin F_d$ ($1 \leq i < p$) and $\text{root}(M_j \downarrow) \in F_d$ ($p \leq j \leq m$) without loss of generality. Let $M_j \downarrow \equiv \hat{C}_j[N_{j,1}, \dots, N_{j,n_j}]$ ($p \leq j \leq m$). Then, by using the induction hypothesis, $M_i^d \downarrow \equiv x$ ($1 \leq i < p$) and $M_j^d \downarrow \equiv \hat{C}_j[x, \dots, x]$ ($p \leq j \leq m$). Thus $M \downarrow \equiv C[M_1 \downarrow, \dots, M_m \downarrow] \downarrow \equiv C[M_1 \downarrow, \dots, M_{p-1} \downarrow, \hat{C}_p[N_{p,1}, \dots, N_{p,n_p}], \dots, \hat{C}_m[N_{m,1}, \dots, N_{m,n_m}]] \downarrow$ and $M^d \downarrow \equiv C[M_1^d \downarrow, \dots, M_m^d \downarrow] \downarrow \equiv C[x, \dots, x, \hat{C}_p[x, \dots, x], \dots, \hat{C}_m[x, \dots, x]] \downarrow$. Note that $M_i \downarrow$ ($1 \leq i < p$), $N_{p,1}, \dots, N_{m,n_m}$ are normal forms having root symbols not in F_d . Therefore, if $\text{root}(M \downarrow) \notin F_d$ then $C[x, \dots, x, \hat{C}_p[x, \dots, x], \dots, \hat{C}_m[x, \dots, x]] \downarrow \equiv x$; if $\text{root}(M \downarrow) \in F_d$ then we have a context

$\hat{C}[\dots,] \equiv C[\dots, \hat{C}_p[\dots,], \dots, \hat{C}_m[\dots,]]$ such that $M \downarrow \equiv \hat{C}[N_1, \dots, N_n]$ where $N_i \in \{M_1 \downarrow, \dots, M_{p-1} \downarrow, N_{p,1}, \dots, N_{m,n_m}\}$ and $M^d \downarrow \equiv \hat{C}[x, \dots, x] \not\equiv x$.

Case 2. $\text{root}(M) \notin F_d$.

Consider three subcases:

Case 2.1. $E_d(M) = \phi$.

From Lemma 3.6, $\text{root}(M \downarrow) \notin F_d$. Since $M^d \equiv x$, $M^d \downarrow \equiv x$.

Case 2.2. $E_d(M) = \{P\}$.

Then $M^d \equiv P^d$. Note that $\text{rank}(P) < k$. Since $M \downarrow \equiv P \downarrow$ and $M^d \downarrow \equiv P^d \downarrow$, the claim follows by using the induction hypothesis.

Case 2.3. $E_d(M) = \{P_1, \dots, P_p\}$ ($p > 1$).

Note that $\text{rank}(P_i) < k$ for any i . From Lemma 3.7, $\text{root}(M \downarrow) \notin F_d$. Since $M \downarrow \equiv P_i \downarrow$, it is clear that $\text{root}(P_i \downarrow) \notin F_d$ for all i . Thus, we have $P_i^d \downarrow \equiv x$ by the induction hypothesis. From case (5) in the definition of M^d , it follows that $M^d \downarrow \equiv x$. \square

Note. Let $E_d(M) = \{P_1, \dots, P_p\}$ ($p > 1$). Then, from Lemma 3.7 and Lemma 4.1, it follows that every P_i^d is erasable. Hence case (6) in the definition of M^d can be removed.

Lemma 4.2. If $P \in E_d(M)$ then $M^d \xrightarrow{*} P^d$.

Proof. Obvious from the definition of M^d and the above note. \square

We wish to translate directly an infinite reduction $M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ into an infinite reduction $M_0^d \xrightarrow{*} M_1^d \xrightarrow{*} M_2^d \xrightarrow{*} \dots$. However, the following example shows that $M_i \rightarrow M_{i+1}$ cannot be translated into $M_i^d \xrightarrow{*} M_{i+1}^d$ in general.

Example 4.1. Consider the two systems R_0 and R_1 :

$$R_0 \quad \left\{ \begin{array}{l} F(C, x) \rightarrow x \\ F(x, C) \rightarrow x \end{array} \right.$$

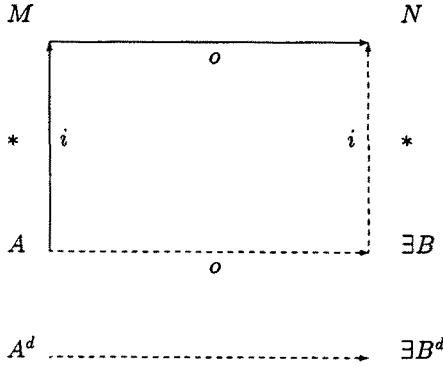
$$R_1 \quad \left\{ \begin{array}{l} f(x) \rightarrow g(x) \\ f(x) \rightarrow h(x) \\ g(x) \rightarrow x \\ h(x) \rightarrow x \end{array} \right.$$

Let $M \equiv F(f(C), h(C)) \rightarrow N \equiv F(g(C), h(C))$. Then $E_1(M) = \{f(C)\}$ and $E_1(N) = \{g(C), h(C)\}$. Thus $M^1 \equiv f(x)$, $N^1 \equiv g(h(x))$. It is obvious that $M^1 \xrightarrow{*} N^1$ does not hold. \square

Now we will consider to translate indirectly an infinite reduction of $R_0 \oplus R_1$ into an infinite reduction of R_d .

We write $M \equiv_o N$ when M and N have the same outermost-layer context, i.e., $M \equiv C[M_1, \dots, M_m]$ and $N \equiv C[N_1, \dots, N_m]$ for some M_i, N_i .

Lemma 4.3. Let $A \xrightarrow{i}^* M$, $M \rightarrow_o N$, $A \equiv_o M$, and $root(M), root(N) \in F_d$. Then, for any A^d there exist B and B^d such that



Proof. Let $A \equiv C[A_1, \dots, A_m]$, $M \equiv C[M_1, \dots, M_m]$, $N \equiv C'[M_{i_1}, \dots, M_{i_n}]$ ($i_j \in \{1, \dots, m\}$). Take $B \equiv C'[A_{i_1}, \dots, A_{i_n}]$. Then, we can obtain $A \rightarrow_o B$ and $B \xrightarrow{i}^* N$. From $A^d \equiv C[A_1^d, \dots, A_m^d]$ and $B^d \equiv C'[A_{i_1}^d, \dots, A_{i_n}^d]$, it follows that $A^d \rightarrow B^d$. \square

Lemma 4.4. Let $M \xrightarrow{*} N$, $root(N) \in F_d$. Then, for any M^d there exist A ($A \equiv_o N$) and A^d such that

Now, take $A \equiv C[x, \dots, x, A_p, \dots, A_{q-1}, M_q, \dots, M_m]$. It is obvious that $M \xrightarrow{*} A$. From Lemma 2.3, we can have the reductions $M_j \xrightarrow{*} N_j$ ($q \leq j \leq m$) in which every term has a root symbol in F_d . Thus it follows that $A \xrightarrow{*}_i N$ and $A \equiv_{\circ} N$. From Lemma 4.1 and $M_i \downarrow \equiv x$ ($1 \leq i < p$), $M_i^d \downarrow \equiv x$. Therefore, since

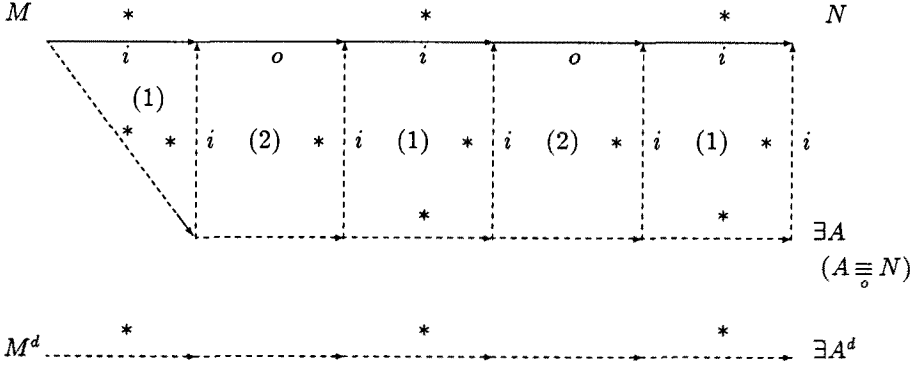
$$M^d \equiv C[M_1^d, \dots, M_{p-1}^d, M_p^d, \dots, M_{q-1}^d, M_q^d, \dots, M_m^d]$$

and $A^d \equiv C[x, \dots, x, A_p^d, \dots, A_{q-1}^d, M_q^d, \dots, M_m^d]$, it follows that $M^d \xrightarrow{*} A^d$. (*end of the claim*)

Now we will prove the lemma for $\text{rank}(M) = k$. Consider two cases.

Case 1. $\text{root}(M) \in F_d$.

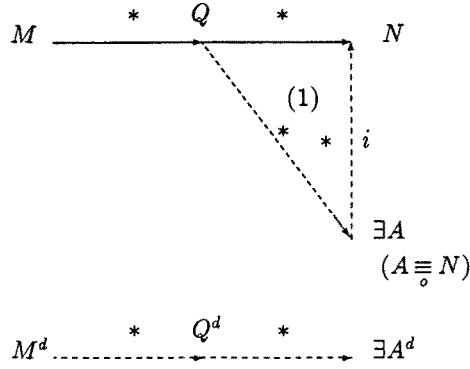
From Lemma 2.3, we may assume that every term in the reduction $M \xrightarrow{*} N$ has a root symbol in F_d . By splitting $M \xrightarrow{*} N$ into $M \xrightarrow{*}_i \xrightarrow{*}_o \xrightarrow{*}_i \xrightarrow{*}_o \cdots \xrightarrow{*}_i N$ and using the claim for diagram (1) and Lemma 5.1 for diagram (2), we can draw the following diagram:



Note that if $M' \xrightarrow{*}_i M'' \xrightarrow{*}_i M'''$ then $M' \xrightarrow{*}_i M'''$; thus, the claim can be applied to diagram (1) in the above diagram.

Case 2. $\text{root}(M) \in F_{\bar{d}}$.

Then we have some essential subterm $Q \in E_d(M)$ such that $M \xrightarrow{*} Q \xrightarrow{*} N$. From Lemma 4.2, it follows that $M^d \xrightarrow{*} Q^d$. It is obvious that $\text{rank}(Q) < k$. Hence, we have the following diagram, where diagram (1) is obtained by the induction hypothesis:



□

Now we can prove the following theorem:

Theorem 4.1. No term M has an infinite reduction.

Proof. We will prove the theorem by induction on $\text{rank}(M)$. The case $\text{rank}(M) = 1$ is trivial. Assume the theorem for $\text{rank}(M) < k$. Then, we will show the case $\text{rank}(M) = k$. Suppose M has an infinite reduction $M \rightarrow \rightarrow \rightarrow \dots$. From the induction hypothesis, we can have no infinite inner reduction $\rightarrow_i \rightarrow_i \rightarrow_i \dots$ in this reduction. Thus, \rightarrow_o must infinitely appear in the infinite reduction. From the induction hypothesis, all of the terms appearing in this reduction have the same rank; hence, their root symbols are in F_d if $\text{root}(M) \in F_d$. Hence, by a similar construction of diagrams as for Case 1 in the proof of Lemma 4.4, it follows that M^d has an infinite reduction. This contradicts that R_d is terminating. □

Corollary 4.1. Two term rewriting systems R_0 and R_1 are left-linear and complete iff the direct sum $R_0 \oplus R_1$ is so.

Proof. \Leftarrow is trivial. \Rightarrow follows from Theorem 4.1 and the theorem in Toyama [8] stating that two term rewriting systems R_0 and R_1 are confluent iff the direct sum $R_0 \oplus R_1$ is so. □

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